Bound of the Derivative and the Polar Derivative of some Classes of Polynomials

ขอบเขตของอนุพันธ์และอนุพันธ์เชิงขั้วของพหุนามบางประเภท

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ABSTRACT

This paper, we consider the following polynomial of degree n:

$$p(z) = (z - z_0)^{t_0}(z - z_1)^{t_1} \cdots (z - z_m)^{t_m}\left(a_0 + \sum_{v=\mu}^{n-(t_0+t_1+\cdots+t_m)} a_v z^v\right)$$

where $a_0, a_v \ (v = \mu, \mu + 1, \dots, n - (t_0 + t_1 + \cdots + t_m)), z_j \ (j = 0,1, \dots, m)$ are complex numbers and $t_0, t_1, \dots, t_m$ are nonnegative integers. We present a lower bound of the derivative of $p(z)$ under the assumption that the zeros $z_0, \dots, z_m$ are outside the closed disc $\{z : |z| \leq k\}$ and the remaining $n - (t_0 + t_1 + \cdots + t_m)$ zeros are inside the disc $\{z : |z| < k\}$ where $k \leq 1$. We also present a lower bound of the polar derivative of $p(z)$ in the situation that $m = 0$ and at least one zero of $p(z)$ belongs to the circle $\{z : |z| = k\}$.

บทคัดย่อ

บทความนี้เราพิจารณาพหุนามระดับขั้น n

$$p(z) = (z - z_0)^{t_0}(z - z_1)^{t_1} \cdots (z - z_m)^{t_m}\left(a_0 + \sum_{v=\mu}^{n-(t_0+t_1+\cdots+t_m)} a_v z^v\right)$$

เมื่อ $a_0, a_v \ (v = \mu, \mu + 1, \dots, n - (t_0 + t_1 + \cdots + t_m)), z_j \ (j = 0,1, \dots, m)$ เป็นจำนวนเชิงซ้อนและ $t_0, t_1, \dots, t_m$ เป็นจำนวนเต็มที่ไม่เป็นลบ เรานำเสนอขอบเขตล่างของอนุพันธ์ของ $p(z)$ ภายใต้ข้อกำหนดคือศูนย์ $z_0, \dots, z_m$ อยู่ภายนอกแผ่นกลมปิด $\{z : |z| \leq k\}$ และศูนย์ที่เหลือ $n - (t_0 + t_1 + \cdots + t_m)$ ตัวอยู่ภายในแผ่นกลม $\{z : |z| < k\}$ เมื่อ $k \leq 1$ นอกจากนี้เรานำเสนอขอบเขตล่างของอนุพันธ์เชิงขั้วของ $p(z)$ ในสถานการณ์ที่ $m = 0$ และมีศูนย์อย่างน้อยหนึ่งตัวอยู่บนวงกลม $\{z : |z| = k\}$

Keywords: Polar derivative, Polynomial, Inequality

คำสำคัญ: อนุพันธ์เชิงขั้ว พหุนาม อสมการ

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Introduction

For a positive real number k, we let $D(0,k)$ and $\overline{D(0,k)}$ denote the set $\{z : |z| < k\}$ and $\{z : |z| \leq k\}$, respectively.

Let $p(z)$ be a polynomial of degree n. According to the well- known Bernstein's inequality (Bernstein, 1926) on the derivative of a polynomial, we have that

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1}$$

Equality holds in (1) if and only if $p(z)$ has all of its zeros at the origin.

If we specialize in the class of the polynomials which have all zeros in $\overline{D(0,1)}$, then it was proved by Turán (1939) that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{2}$$

The inequality (2) is best possible with equality for $p(z) = az^n + b$ with $|a| = |b|$.

The inequality (2) under the same hypothesis was refined by Aziz, Dawood (1988). They proved that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2}\left[\max_{|z|=1} |p(z)| + \min_{|z|=1} |p(z)|\right]. \tag{3}$$

This result is best possible and equality in (3) holds for $p(z) = az^n + b$ with $|b| \leq |a|$.

For the class of polynomials $p(z)$ of degree n having all zeros in $\overline{D(0,k)}, k \leq 1$, Govil (1991) proved that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k}\left[\max_{|z|=1} |p(z)| + \frac{1}{k^{n-1}} \min_{|z|=k} |p(z)|\right]. \tag{4}$$

The inequality (4) is best possible and equality holds for $p(z) = (z+k)^n$.

Aziz, Shah (1997) proved that if $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v,\ 1 \leq \mu \leq n$, is a polynomial of degree n having all zeros in $\overline{D(0,k)}, k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^\mu}\left[\max_{|z|=1} |p(z)| + \frac{1}{k^{n-\mu}} \min_{|z|=k} |p(z)|\right]. \tag{5}$$

The inequality (5) is best possible and equality holds for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

Somsuwan and Nakprasit (2013) investigated the polynomial of degree n which have zero z_0 with $|z_0| > k$ and other zeros are inside $D(0,k), k \leq 1$. One of their results is as follows.

Theorem 1.1 (Somsuwan, Nakprasit, 2013) Let $k \leq 1$ and $p(z) = (z - z_0)^s\left(a_0 + \sum_{v=\mu}^{n-s} a_v z^v\right), 1 \leq \mu \leq n - s$, be a polynomial of degree n having zero z_0 outside $\overline{D(0,k)}$ and the remaining $n - s$ zeros are in $D(0,k)$. Then

$$\max_{|z|=1} |p'(z)| \geq \left[\frac{A}{(1+|z_0|)^s} - \frac{s}{(1+|z_0|)}\right] \max_{|z|=1} |p(z)| + \left[\frac{A}{k^{n-s-\mu}(k+|z_0|)^s}\right] \min_{|z|=k} |p(z)|,$$

where $A = \frac{\left|1-|z_0|\right|^s (n-s)}{(1+k^\mu)}$.

The *polar derivative* of a polynomial $p(z)$ of degree n with respect to a complex number α, denoted by $D_\alpha p(z)$, is defined by

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

Note that the polynomial $D_\alpha p(z)$ has degree at most $n-1$ and generalizes the derivative of polynomial in sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha p(z)}{\alpha} = p'(z).$$

The lower bound of $D_\alpha p(z)$ have been studied by many researchers. For example, Shah (1996), Aziz, Rather (1998), Govil, Mctume (2004), and Dewan et al. (2009) studied a lower bound of $\max_{|z|=1} |D_\alpha p(z)|$ where $p(z)$ is a polynomial of degree n having all zeros in $\overline{D(0,k)}, k \leq 1$ and $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$.

Lower bound of a derivative of a polynomial

Let $k \leq 1$. In this section, we investigate a lower bound of $\max_{|z|=1} |p'(z)|$ where $p(z)$ is a polynomial of degree n having zeros $z_0, \dots, z_m$ are outside $\overline{D(0,k)}$ and the remaining $n-(t_0+\cdots+t_m)$ zeros are in $D(0,k)$.

Theorem 2.1 Let $k \leq 1$ and $p(z)$ be a polynomial of degree n in the form

$$p(z) = (z-z_1)^{t_1}(z-z_0)^{t_0}\left(a_0 + \sum_{v=\mu}^{n-(t_1+t_0)} a_v z^v\right), 1 \leq \mu \leq n-(t_1+t_0).$$

If zeros z_0 and z_1 of $p(z)$ are outside $\overline{D(0,k)}$ and the remaining $n-(t_1+t_0)$ zeros are in $D(0,k)$. Then

$$\max_{|z|=1} |p'(z)| \geq \left[\frac{\left|1-|z_1|\right|^{t_1} A}{(1+|z_0|)^{t_0}(1+|z_1|)^{t_1}} - \frac{t_0\left|1-|z_1|\right|^{t_1}}{(1+|z_0|)(1+|z_1|)^{t_1}} - \frac{t_1}{(1+|z_1|)}\right] \max_{|z|=1} |p(z)|$$
$$+ \left[\frac{\left|1-|z_1|\right|^{t_1} A}{k^{n-(t_1+t_0)-\mu}(k+|z_0|)^{t_0}(k+|z_1|)^{t_1}}\right] \min_{|z|=k} |p(z)|,$$

where $A = \frac{\left|1-|z_0|\right|^{t_0}(n-(t_1+t_0))}{1+k^\mu}$.

Proof. Let $p(z) = (z-z_1)^{t_1} p_0(z)$ where $p_0(z) = (z-z_0)^{t_0}\left(a_0 + \sum_{v=\mu}^{n-(t_1+t_0)} a_v z^v\right)$.

The derivative of $p(z)$ is

$$p'(z) = (z-z_1)^{t_1} p'_0(z) + t_1 (z-z_1)^{t_1-1} p_0(z).$$

The triangle inequality implies that

$$|p'(z)| \geq |z-z_1|^{t_1} |p'_0(z)| - t_1 |z-z_1|^{t_1-1} |p_0(z)|.$$

That is,

$$|p'(z)| + t_1 |z-z_1|^{t_1-1} |p_0(z)| \geq |z-z_1|^{t_1} |p'_0(z)|.$$

Consequently,

$$\max_{|z|=1} |p'(z)| + \max_{|z|=1} \left(t_1 |z-z_1|^{t_1-1} |p_0(z)|\right) \geq \max_{|z|=1} |z-z_1|^{t_1} |p'_0(z)|.$$

That is,

$$\max_{|z|=1} |p'(z)| \geq \max_{|z|=1} |z-z_1|^{t_1} |p'_0(z)| - \max_{|z|=1} \left(t_1 |z-z_1|^{t_1-1} |p_0(z)|\right).$$

Since $|z-z_1|^{t_1} \geq (|z|-|z_1|)^{t_1} = (1-|z_1|)^{t_1}$ for $k < |z_1| < 1$ and $|z-z_1|^{t_1} = |z_1 - z|^{t_1} \geq (|z_1|-|z|)^{t_1}$ $= (|z_1|-1)^{t_1}$ for $|z_1| > 1$, we obtain that $|z-z_1|^{t_1} \geq \left|1-|z_1|\right|^{t_1}$ for $|z_1| > k$.

For $|z| = 1$, we get that $(|z-z_1|)^{t_1-1} \leq (|z|+|z_1|)^{t_1-1} = (1+|z_1|)^{t_1-1}$.

By combining these two results, we get that

$$\max_{|z|=1} |p'(z)| \geq \left|1-|z_1|\right|^{t_1} \max_{|z|=1} |p'_0(z)| - t_1 (1+|z_1|)^{t_1-1} \max_{|z|=1} |p_0(z)|.$$

By applying $p_0(z)$ in Theorem 1.1, we obtain that

$$\max_{|z|=1} |p'_0(z)| \geq \left[\frac{A}{(1+|z_0|)^{t_0}} - \frac{t_0}{(1+|z_0|)}\right] \max_{|z|=1} |p_0(z)| + \left[\frac{A}{k^{n-(t_1+t_0)-\mu}(k+|z_0|)^{t_0}}\right] \min_{|z|=k} |p_0(z)|,$$

where $A = \frac{\left|1-|z_0|\right|^{t_0}(n-(t_1+t_0))}{1+k^\mu}$.

Therefore,

$$\max_{|z|=1} |p'(z)| \geq \left[\frac{\left|1-|z_1|\right|^{t_1} A}{(1+|z_0|)^{t_0}} - \frac{\left|1-|z_1|\right|^{t_1} t_0}{(1+|z_0|)} - t_1 (1+|z_1|)^{t_1-1}\right] \max_{|z|=1} |p_0(z)|$$
$$+ \left[\frac{\left|1-|z_1|\right|^{t_1} A}{k^{n-(t_1+t_0)-\mu}(k+|z_0|)^{t_0}}\right] \min_{|z|=k} |p_0(z)|.$$

Since $|p_0(z)| = \frac{1}{|z-z_1|^{t_1}} \cdot |p(z)| \geq \frac{1}{(1+|z_1|)^{t_1}} \cdot |p(z)|$ for $|z| = 1$,

$$\max_{|z|=1}|p_0(z)| = \max_{|z|=1}\left[\frac{1}{|z-z_1|^{t_1}}\cdot|p(z)|\right] \geq \frac{1}{(1+|z_1|)^{t_1}}\cdot\max_{|z|=1}|p(z)|.$$

For $|z| = k$, we have $|p_0(z)| = \frac{1}{|z-z_1|^{t_1}}\cdot|p(z)| \geq \frac{1}{(k+|z_1|)^{t_1}}\cdot|p(z)|$ and then

$$\min_{|z|=k}|p_0(z)| = \min_{|z|=k}\left[\frac{1}{|z-z_1|^{t_1}}\cdot|p(z)|\right] \geq \frac{1}{(k+|z_1|)^{t_1}}\cdot\min_{|z|=k}|p(z)|.$$

Consequently,

$$\max_{|z|=1}|p'(z)| \geq \left[\frac{\left|1-|z_1|\right|^{t_1}A}{(1+|z_0|)^{t_0}(1+|z_1|)^{t_1}} - \frac{\left|1-|z_1|\right|^{t_1}t_0}{(1+|z_0|)(1+|z_1|)^{t_1}} - \frac{t_1}{(1+|z_1|)}\right]\max_{|z|=1}|p(z)|$$
$$+\left[\frac{\left|1-|z_1|\right|^{t_1}A}{k^{n-(t_1+t_0)-\mu}(k+|z_0|)^{t_0}(k+|z_1|)^{t_1}}\right]\min_{|z|=k}|p(z)|,$$

where $A = \frac{\left|1-|z_0|\right|^{t_0}(n-(t_1+t_0))}{1+k^{\mu}}$.

Remark Consider a polynomial of degree n

$$p(z) = (z-z_m)^{t_m}(z-z_{m-1})^{t_{m-1}}\cdots(z-z_0)^{t_0}\left(a_0 + \sum_{v=\mu}^{n-(t_m+\cdots+t_0)} a_v z^v\right)$$

where zero $z_0,\dots,z_m$ are outside $\overline{D(0,k)}$, $k \leq 1$ and the remaining $n-(t_m+\cdots+t_0)$ zeros are in $D(0,k), k \leq 1$.

A lower bound of $\max\limits_{|z|=1}|p'(z)|$ can be obtained by applying Theorem 1.1 as in the proof of Theorem 2.1.

Let $p_0(z) = (z-z_0)^{t_0}\left(a_0 + \sum_{v=\mu}^{n-(t_m+\cdots+t_0)} a_v z^v\right), p_j(z) = (z-z_j)^{t_j}p_{j-1}(z)$ for $1 \leq j \leq m$. Theorem 1.1 yields a lower bound of $\max\limits_{|z|=1}|p'_0(z)|$. Combining this lower bound together with the facts that

$$\max_{|z|=1}|p_0(z)| \geq \frac{1}{(1+|z_1|)^{t_1}}\cdot\max_{|z|=1}|p_1(z)| \text{ and } \min_{|z|=k}|p_0(z)| \geq \frac{1}{(k+|z_1|)^{t_1}}\cdot\min_{|z|=k}|p_1(z)|.$$

The lower bound of $\max\limits_{|z|=1}|p'_1(z)|$ can be obtained as in Theorem 2.1.

Consequently, a lower bound of $\max\limits_{|z|=1}\left|p'_j(z)\right|$ for $2 \leq j \leq m$ can be obtained by similar process by using a lower bound of $\max\limits_{|z|=1}\left|p'_{j-1}(z)\right|$ and the facts that

$$\max_{|z|=1}\left|p_{j-1}(z)\right| \geq \frac{1}{(1+|z_j|)^{t_j}}\cdot\max_{|z|=1}\left|p_j(z)\right| \text{ and } \min_{|z|=k}\left|p_{j-1}(z)\right| \geq \frac{1}{(k+|z_j|)^{t_j}}\cdot\min_{|z|=k}\left|p_j(z)\right| \text{ for } 2 \leq j \leq m.$$

Lower bound of a polar derivative of a polynomial

Let $k \leq 1$. In this section, we study a lower bound of the polar derivative of polynomial in the form

$$p(z) = (z-z_0)^s\left(a_0 + \sum_{v=\mu}^{n-s} a_v z^v\right), 1 \leq \mu \leq n-s$$

having zero z_0 outside $\overline{D(0,k)}$ and the remaining $n-s$ zeros are in $\overline{D(0,k)}$ with at least one zero on $\{z : |z| = k\}$.

Theorem 3.1 Let $C(0,k) = \{z : |z| = k\}$ and $p(z)$ be a polynomial of degree n in the form

$$p(z) = (z-z_0)^s\left(a_0 + \sum_{v=\mu}^{n-s} a_v z^v\right), 1 \leq \mu \leq n-s.$$

Let $k \leq 1$ and $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$. If a zero z_0 of $p(z)$ is outside $\overline{D(0,k)}$ and the remaining $n-s$ zeros are in $\overline{D(0,k)}$ with at least one zero on $C(0,k)$, then

$$\max_{|z|=1}|D_\alpha p(z)| \geq \left[\frac{(|\alpha|-1)A}{(1+|z_0|)^s} - \left(n + \frac{s(|\alpha|+1)}{(1+|z_0|)}\right)\right]\max_{|z|=1}|p(z)|,$$

where $A = \frac{\left|1-|z_0|\right|^s (n-s)}{(1+k^\mu)}$.

Proof. By setting $\phi(z) = a_0 + \sum_{v=\mu}^{n-s} a_v z^v$, we can rewrite $p(z) = (z - z_0)^s \phi(z)$.

The derivative of $p(z)$ is $p'(z) = (z - z_0)^s \phi'(z) + s(z - z_0)^{s-1}\phi(z)$ and then

$$D_\alpha p(z) = (\alpha - z)(z - z_0)^s \phi'(z) + [n(z - z_0) + s(\alpha - z)](z - z_0)^{s-1}\phi(z).$$

The triangle inequality implies that

$$|D_\alpha p(z)| + |[n(z - z_0) + s(\alpha - z)](z - z_0)^{s-1}\phi(z)| \geq |(\alpha - z)(z - z_0)^s \phi'(z)|.$$

One can see that

$$\max_{|z|=1}|D_\alpha p(z)| \geq \max_{|z|=1}|(\alpha - z)(z - z_0)^s \phi'(z)| - \max_{|z|=1}|[n(z - z_0) + s(\alpha - z)](z - z_0)^{s-1}\phi(z)|.$$

For $|z| = 1$, we get that $|z - z_0| \leq |z| + |z_0| = 1 + |z_0|$ and $|\alpha| - 1 = |\alpha| - |z| \leq |\alpha - z| \leq |\alpha| + |z| = |\alpha| + 1$. Since $|z - z_0|^s \geq (|z| - |z_0|)^s = (1 - |z_0|)^s$ for $k < |z_0| < 1$ and $|z - z_0|^s = |z_0 - z|^s \geq (|z_0| - |z|)^s = (|z_0| - 1)^s$ for $|z_0| > 1$, we obtain that $|z - z_0|^s \geq \left|1 - |z_0|\right|^s$ for $|z_0| > k$.

Consequently,

$$\max_{|z|=1}|D_\alpha p(z)| \geq (|\alpha| - 1)\left|1 - |z_0|\right|^s \max_{|z|=1}|\phi'(z)| - [n(1 + |z_0|) + s(|\alpha| + 1)](1 + |z_0|)^{s-1}\max_{|z|=1}|\phi(z)|.$$

Because $p(z)$ has a zero on $C(0,k)$, also $\phi(z)$ has a zero on $C(0,k)$ and $\min_{|z|=k}|\phi(z)| = 0$.

By applying $\phi(z)$ in the inequality (5), we have that

$$\max_{|z|=1}|\phi'(z)| \geq \frac{n-s}{1+k^\mu} \cdot \max_{|z|=1}|\phi(z)|.$$

This implies that

$$\max_{|z|=1}|D_\alpha p(z)| \geq \left[\frac{(|\alpha| - 1)\left|1 - |z_0|\right|^s (n-s)}{1+k^\mu} - [n(1 + |z_0|) + s(|\alpha| + 1)](1 + |z_0|)^{s-1}\right]\max_{|z|=1}|\phi(z)|.$$

Observe that $|\phi(z)| = \frac{1}{|z-z_0|^s} \cdot |p(z)| \geq \frac{1}{(1+|z_0|)^s} \cdot |p(z)|$ for $|z| = 1$.

Then

$$\max_{|z|=1}|\phi(z)| = \max_{|z|=1}\left[\frac{1}{|z - z_0|^s} \cdot |p(z)|\right] \geq \frac{1}{(1 + |z_0|)^s} \cdot \max_{|z|=1}|p(z)|.$$

Therefore,

$$\begin{aligned}
\max_{|z|=1}|D_\alpha p(z)| &\geq \left[\frac{(|\alpha| - 1)\left|1 - |z_0|\right|^s (n-s)}{1+k^\mu} - [n(1 + |z_0|) + s(|\alpha| + 1)](1 + |z_0|)^{s-1}\right] \\
&\quad \times \left[\frac{1}{(1+|z_0|)^s} \cdot \max_{|z|=1}|p(z)|\right] \\
&= \left[\frac{(|\alpha| - 1)\left|1 - |z_0|\right|^s (n-s)}{(1+k^\mu)(1 + |z_0|)^s} - \left(n + \frac{s(|\alpha| + 1)}{(1 + |z_0|)}\right)\right]\max_{|z|=1}|p(z)| \\
&= \left[\frac{(|\alpha| - 1)A}{(1 + |z_0|)^s} - \left(n + \frac{s(|\alpha| + 1)}{(1 + |z_0|)}\right)\right]\max_{|z|=1}|p(z)|,
\end{aligned}$$

where $A = \frac{\left|1-|z_0|\right|^s (n-s)}{(1+k^\mu)}$.

Conclusions

This paper give a lower bound of a derivative of a polynomial of degree n:

$$p(z) = (z - z_0)^{t_0}(z - z_1)^{t_1} \cdots (z - z_m)^{t_m}\left(a_0 + \sum_{v=\mu}^{n-(t_0+t_1+\cdots+t_m)} a_v z^v\right)$$

where $a_0, a_v\ (v = \mu, \mu + 1, \ldots, n - (t_0 + t_1 + \cdots + t_m)), z_j\ (j = 0,1,\ldots,m)$ are complex numbers and $t_0, t_1, \ldots, t_m$ are nonnegative integers under the assumption that the zeros $z_0, \ldots, z_m$ are outside the closed disc $\overline{D(0,k)}$ and the remaining $n - (t_0 + t_1 + \cdots + t_m)$ zeros are inside the disc $D(0,k)$ where $k \leq 1$ and also give a lower bound of the polar derivative of $p(z)$ in the situation that $m = 0$ and at least one zero of $p(z)$ belongs to the circle $\{z : |z| = k\}$.

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