Characterizations of Regular Ordered Semirings by Generalizations of Ordered Ideals

การจำแนกลักษณะเฉพาะของกึ่งริงอันดับปรกติโดยรูปแบบทั่วไปของไอดีลอันดับ

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ABSTRACT

The purposes of this study are to study ordered m-right ideals, ordered m-left ideals, ordered (m,n)-quasi ideals and define the notion of an ordered m-bi-ideals of an ordered semiring and use these ordered ideals to characterize regular ordered semirings. Moreover, we found an interesting result that an ordered semiring S is regular if and only if there exist natural numbers m,n such that $(\Sigma RL] = R \cap L$ or $B = (\Sigma BS^mB]$ for every ordered m-right ideal R, every ordered n-left ideal L and every ordered m-bi-ideal B of S.

บทคัดย่อ

การศึกษานี้มีวัตถุประสงค์เพื่อศึกษา ไอดีล m-ขวาอันดับ ไอดีล m-ซ้ายอันดับ (m,n)-ควอซีไอดีลอันดับ และนิยาม m-bi ไอดีลอันดับของกึ่งริงอันดับ และนำไอดีลอันดับข้างต้นไปใช้จำแนกลักษณะเฉพาะของกึ่งริงอันดับปรกติ ยิ่งกว่านั้นเราพบผลการศึกษาที่น่าสนใจว่า กึ่งริงอันดับ S จะเป็นกึ่งริงอันดับปรกติก็ต่อเมื่อมีจำนวนนับ m และ n ที่ซึ่ง $(\Sigma RL] = R \cap L$ หรือ $B = (\Sigma BS^mB]$ สำหรับทุกไอดีล m-ขวาอันดับ R ทุกไอดีล n-ซ้ายอันดับ L และทุก m-ไบไอดีลอันดับ B ของ S

Keywords: Ordered semiring, Ordered m-bi-ideal, Regular ordered semiring

คำสำคัญ: กึ่งริงอันดับ m-ไบไอดีลอันดับ กึ่งริงอันดับปรกติ

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Introduction

The notion of a quasi-ideal of a ring was defined by Steinfeld (1978) then he studied some of its properties. Later, Dönges (1994) considered quasi-ideals of semirings with an absorbing zero element, studied some of their properties and investigated the connections between them and other ideals of semirings. Then Shabir, Ali and Batool (2004) studied some more properties of quasi ideals and used them to characterize regular semirings. Afterward, Chinram (2008) defined a generalization of a quasi-ideal of a semiring called an (m,n)-quasi-ideal, investigated its properties and use it to characterize a regular semiring.

An ordered semiring which introduced first by Gan and Jiang (2011) is an interesting generalization of semiring as a semiring together with a partially ordered relation connected by the compatibility property. In (Gan & Jiang, 2011), the concept of a left (right) ordered ideal was also defined. Then Mandal, 2014 defined regular ordered semirings and considered fuzzy ideals of ordered semirings. Later, Palakawong na Ayutthaya and Pibaljommee (2016) introduced the concept of ordered quasi-ideals of ordered semirings, studied some of their properties, investigated connections between them and other ordered ideals and use them to characterize regular, left regular and right regular ordered semirings.

In 2017, Omidi and Davvaz generalized the notion of ordered left (right) ideals and ordered qausi-ideals of ordered semirings to be m-left (right) hyperideal and (m,n)-quasi-hyperideals, resp., of ordered semihyperings.

A new concept of a general form of a bi-ideal of a semiring called an m-bi-ideal was presented by Munir and Shafiq (2018) and its important properties from the pure algebraic point of view have been described. Moreover, Munir and Shafiq presented the form of the m-bi-ideal generated by a nonempty subset of a semiring.

The purpose of this study is to generalize the work of Palakawong na Ayutthaya and Pibaljommee (2016) and Munir and Shafiq (2018). It means that we study an ordered m-right ideal, an ordered m-left ideal, an ordered (m,n)-quasi-ideal and define the notion of an ordered m-bi-ideal of an ordered semiring. Then we give some characterizations of regular ordered semirings by ordered m-right ideals, ordered m-left ideals and ordered m-bi-ideals.

Objectives of the study

1) To study an ordered m-right ideals, ordered m-left ideals, ordered (m,n)-quasi-ideals and define the notion of an ordered m-bi-ideal of ordered semirings.

2) To characterize regular ordered semirings using their ordered m-right ideals, ordered m-left ideals and ordered m-bi-ideals.

Preliminaries

In this section, we recall some necessary definitions, notations and properties of some algebraic structures.

Throughout this work, we let $\mathbb{N}$ be the set of all positive integers.

A *semiring* $(S,+,\cdot)$ is a tri-tuple consisting of a nonempty set S, two binary operations $+$ and $\cdot$ on S such that $(S,+)$ and $(S,\ \cdot)$ are semigroups which are connected by the distributive law. From now on, we shall simply write ab instead of $a\cdot b$ for all $a,b\in S$. A nonempty subset T of S is said to be a *subsemiring* of S if T is a semiring with respect to the same binary operations of S. A semiring S is called *additively commutative* if $a+b=b+a$ for any $a,b\in S$. An element 0 of a semiring S is called an *absorbing zero* if $0+x=x=x+0$ and $0x=0=x0$ for all $x\in S$.

Let A and B be nonempty subsets of a semiring S with absorbing zero 0, $a\in S$ and $n\in\mathbb{N}$. Then we denote the following notations:

$A+B=\{a+b\in S \mid a\in A, b\in B\}$;

$AB=\{ab\in S \mid a\in A, b\in B\}$;

$A^n=AA\cdots A$ (n times) and

$\Sigma A=\{\sum_{i\in I} a\in S \mid a\in A \text{ and } I \text{ is a finite subset of } \mathbb{N}\}$.

We write Σa instead of $\Sigma\{a\}$ for any $a\in S$ and note that $\sum_{i\in\emptyset} a_i=0$.

A nonempty subset A of a semiring S is called a *left ideal* (resp. *right ideal*) of S if $A+A\subseteq A$ and $SA\subseteq A$ (resp. $AS\subseteq A$). We call a subsemiring B of S a *bi-ideal* of S if $BSB\subseteq B$. It is easy to see that every left ideal every right ideal of a semiring S is a bi-ideal.

Let $m\in\mathbb{N}$. According to the work of Munir and Shafiq (2018), a subsemiring A of a semiring S an *m-left ideal* (resp. *m-right ideal*) of S if $S^mA\subseteq A$ (resp. $AS^m\subseteq A$). Furthermore, they call a subsemiring B of S an *m-bi-ideal* if $BS^mB\subseteq B$. The power m of an m-bi-ideal B is called *bipotency* of B. Moreover, $BS^mB\subseteq B$ is called the bipotency condition. It is to be noted that a bi-ideal B of a semiring S is a 1-bi ideal of S (a bi-ideal of bipotency 1). In this work, we simply call a bi-ideal instead of a 1-bi-ideal.

Remark 1.1. (Munir & Shafiq, 2018) A For every $m\geq 1$, every bi-ideal is an m-bi-ideal.

The converse of the above remark is not true as is evident from the following example which is given by Munir and Shafiq (2018).

Example 1.2. (Munir & Shafiq, 2018) Let $S=\left\{\begin{bmatrix}0&u&v&w\\0&0&x&y\\0&0&0&z\\0&0&0&0\end{bmatrix}: u,v,w,x,y\in\mathbb{R}^+\right\}\cup\left\{\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\right\}$. Then $(S,+,\cdot)$ is a semiring under the usual operations of addition $+$ and multiplication $\cdot$ of matrices.

Let $B=\left\{\begin{bmatrix}0&u&0&0\\0&0&0&0\\0&0&0&z\\0&0&0&0\end{bmatrix}: u,z\in\mathbb{R}^+\right\}\cup\left\{\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\right\}$. Then B is a 2-bi-ideal of S but not a bi-ideal, since $BSB\nsubseteq B$.

Remark 1.3. (Munir & Shafiq, 2018) Let $m,n\in\mathbb{N}$ such that $m\geq n$. Then every n-bi-ideal of a semiring S is an m-bi-ideal of S.

An algebraic structure $(S,+,\cdot,\leq)$ is called an *ordered semiring* (Gan & Jiang, 2011) if $(S,+,\cdot)$ is a semiring, $(S,\leq)$ is a partially ordered set satisfying the properties; for all $a,b,c \in S$, if $a \leq b$ then $a+c \leq b+c,\ c+a \leq c+b,\ ac \leq bc$ and $ca \leq cb$.

Throughout this study, we always assume that every ordered semiring is additively commutative containing an absorbing zero and also write S instead of $(S,+,\cdot,\leq)$.

We denote the notation $(A] = \{x \in S \mid x \leq a\ \exists a \in A\}$ for all nonempty subsets A of an ordered semiring S.

We recall some necessary basic properties which occur in (Palakawong na Ayutthaya, Pibaljommee, 2016) as follows.

Remark 1.4. Let A and B be nonempty subsets of an ordered semiring S. Then the following statements hold:

(i) $A \subseteq \Sigma A$ and $\Sigma(\Sigma A) = \Sigma A$;

(ii) if $A \subseteq B$ then $\Sigma A \subseteq \Sigma B$;

(iii) $A(\Sigma B) \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$ and $(\Sigma A)B \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$;

(iv) $\Sigma(A+B) \subseteq \Sigma A + \Sigma B$;

(v) $\Sigma(A] \subseteq (\Sigma A]$;

(vi) $A \subseteq (A]$ and $((A]] = (A]$;

(vii) if $A \subseteq B$ then $(A] \subseteq (B]$;

(viii) $A(B] \subseteq (A](B] \subseteq (AB]$ and $(A]B \subseteq (A](B] \subseteq (AB]$;

(ix) $A + (B] \subseteq (A] + (B] \subseteq (A+B]$;

(x) $(A \cup B] = (A] \cup (B]$;

(xi) $(A \cap B] \subseteq (A] \cap (B]$.

According to the work of Gan and Jiang (2011) a nonempty subset A of an ordered semiring S such that $A = (A]$ is called a *left ordered ideal* (resp. *right ordered ideal*) of S if A is a left ideal (resp. right ideal) of S. A nonempty subset Q of S such that $Q = (Q]$ is said to be an *ordered quasi-ideal* of S (Palakawong na Ayutthaya & Pibaljommee, 2016) if $(\Sigma SQ] \cap (\Sigma QS] \subseteq Q$. A subsemiring B of S such that $B = (B]$ is called an *ordered bi-ideal* of S (Palakawong na Ayutthaya & Pibaljommee, 2016) if B is a bi-ideal of S.

According to the work of Mandal (2014), an ordered semiring S to be *regular* if for all $a \in S$ there exists $x \in S$ such that $a \leq axa$.

Remark 1.5. Let S be an ordered semiring. Then the following statements are equivalent:

(i) S is regular;

(ii) $A \subseteq (\Sigma ASA]$ for each $A \subseteq S$;

(iii) $a \in (aSa]$ for each $a \in S$.

Main results

First, we present the definition of an ordered m-left (right) ideals of an ordered semirings which is a special case of Definition 3.2 in (Omidi & Davvaz, 2017).

Definition 2.1. Let S be an ordered semiring, A be a subsemiring of S and $m \in \mathbb{N}$. Then A is said to be an *ordered m-left ideal* (resp. *ordered m-right ideal*) of S if $S^mA \subseteq A$ (resp. $AS^m \subseteq A$) and $A = (A]$.

We note that a left ordered ideal and a right ordered ideal is an ordered 1-left ideal and an ordered 1-right ideal of S, respectively.

Remark 2.2. Let A be a nonempty subset of an ordered semiring S and $m \in \mathbb{N}$. The following properties hold:

(i) $(\Sigma S^mA]$ is an ordered m-left ideal of S;

(ii) $(\Sigma AS^m]$ is an ordered m-right ideal of S.

Proof. (i) Let A be a nonempty subset of an ordered semiring S. It is not difficult to show that $(\Sigma S^mA]$ is closed under addition and multiplication. We have that $S^m(S^mA] \subseteq (S^mS^mA] \subseteq (S^{2m}A] \subseteq (S^mA]$. Clearly, $((S^mA]] = (S^mA]$.

Therefore, $(\Sigma S^mA]$ is an ordered m-left ideal of S.

(ii) can be proved similarly.

Remark 2.3. Let $m, n \in \mathbb{N}$ such that $m \geq n$. Then every ordered n-left (resp. right) ideal of an ordered semiring S is an ordered m-left (resp. right) ideal of S.

Now, we specialize the Definition 3.1 in (Omidi & Davvaz, 2017) to define the notion of an ordered (m, n)-quasi-ideal of an ordered semiring.

Definition 2.4. Let $m, n \in \mathbb{N}$. Then a subsemiring Q of an ordered semiring S is said to be an *ordered (m, n)-quasi-ideal* of S if $(\Sigma S^mQ] \cap (\Sigma QS^n] \subseteq Q$ and $Q = (Q]$.

We note that an ordered quasi-ideal of an ordered semiring S is an ordered $(1,1)$-quasi-ideal of S.

The following remark can be proved in a similar way as proved in Remark 2.2(i).

Remark 2.5. Let A be a nonempty subset of an ordered semiring S and $m, n \in \mathbb{N}$. Then $(\Sigma S^mA] \cap (\Sigma AS^n]$ is an ordered (m, n)-quasi-ideal of S.

Remark 2.6. Let $m, n, s, t \in \mathbb{N}$ such that $m \geq s$ and $n \geq t$. Then every ordered (s, t)-quasi-ideal of an ordered semiring S is an ordered (m, n)-quasi-ideal of S.

Remark 2.7. Let S be an ordered semiring and $m, n \in \mathbb{N}$. Then every ordered m-left ideal of S is an ordered (m, n)-quasi-ideal of S for any $n \geq 1$.

Proof. Let A be an ordered m-left ideal of S. It is sufficient to show that $(\Sigma S^mA] \cap (\Sigma AS^n] \subseteq A$. We obtain that $(\Sigma S^mA] \cap (\Sigma AS^n] \subseteq (\Sigma S^mA] \subseteq (\Sigma A] = (A] = A$. Therefore, A is an ordered (m, n)-quasi-ideal of S.

The following remark can be proved in a similar way of Remark 2.7.

Remark 2.8. Let S be an ordered semiring and $m, n \in \mathbb{N}$. Then every ordered n-right ideal of S is an ordered (m, n)-quasi-ideal of S for any $m \geq 1$.

The converses of Remark 2.7 and 2.8 are not true as show by following two examples.

Example 2.9. Let $S = \{a,b,c,d\}$. Define binary operations $+$ and $\cdot$ on S by the following tables:

$+$	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	b	c	d
d	d	b	d	d

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	c	c	c
d	a	b	b	b

Define a binary relation $\leq$ on S by $\leq := \{(a,a),(b,b),(c,c),(d,d),(b,d)\}$. Then $(S,+,\cdot,\leq)$ is an additively commutative ordered semiring with an absorbing zero a. Let $Q = \{a,b\}$. We have that Q is an ordered $(2,n)$-quasi-ideal of S for any $n \in \mathbb{N}$, but Q is not an ordered 2-left ideal of S.

Example 2.10. Let $S = \{a,b,c,d\}$. Define binary operations $+$ and $\cdot$ on S by the following tables:

$+$	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	b	c	d
d	d	b	d	d

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	b
c	a	b	c	b
d	a	b	c	b

Define a binary relation $\leq$ on S by $\leq := \{(a,a),(b,b),(c,c),(d,d),(b,d)\}$. Then $(S,+,\cdot,\leq)$ is an additively commutative ordered semiring with an absorbing zero a. Let $Q = \{a,b\}$. We have that Q is an ordered $(m,2)$-quasi-ideal of S for any $m \in \mathbb{N}$, but Q is not an ordered 2-right ideal of S.

Definition 2.11. Let B be a subsemiring of an ordered semiring S and $m \in \mathbb{N}$. Then B is said to be an *ordered m-bi-ideal* of S if $BS^mB \subseteq B$ and $B = (B]$.

We note that an ordered bi-ideal of an ordered semiring S is an ordered m-bi-ideal of S, for all $m \in \mathbb{N}$.

Using Definition 2.11, we can easily obtain the following two remarks.

Remark 2.12. Let A be a nonempty subset of an ordered semiring S and $m \in \mathbb{N}$. Then $(\Sigma AS^mA]$ is an ordered m-bi-ideal of S.

Remark 2.13. Let $m,n \in \mathbb{N}$ such that $m \geq n$. Then every ordered n-bi-ideal of an ordered semiring S is an ordered m-bi-ideal of S.

Remark 2.14. Let Λ be a nonempty index set and B_α be an ordered m_α-bi-ideals of an ordered semiring S where $\alpha \in \Lambda$ and $m_\alpha \in \mathbb{N}$. Then $\bigcap_{\alpha\in\Lambda} B_\alpha$ is an ordered $\max\{m_\alpha \mid \alpha \in \Lambda\}$-bi-ideal of S.

Proof. We set the $B = \bigcap_{\alpha\in\Lambda} B_\alpha$. It is sufficient to show that $BS^{\max\{m_\alpha|\alpha\in\Lambda\}}B \subseteq B$ and $B = (B]$ because arbitrary intersection of m-bi-ideal of S is a m-bi-ideal of S. Since $B_\alpha S^{m_\alpha} B_\alpha \subseteq B_\alpha$ for any $\alpha \in \Lambda$, we can obtain that

$$BS^{\max\{m_\alpha|\alpha \in \Lambda\}}B \subseteq B_\alpha S^{\max\{m_\alpha|\alpha \in \Lambda\}}B_\alpha$$

$$\subseteq B_\alpha S^{m_\alpha} B_\alpha$$
$$\subseteq B_\alpha \text{ for any } \alpha \in \Lambda.$$

Thus, $BS^{\max\{m_\alpha | \alpha \in \Lambda\}}B \subseteq \bigcap_{\alpha\in\Lambda} B_\alpha = B$. Finally, we have that

$$B = \bigcap_{\alpha\in\Lambda} B_\alpha \subseteq (\bigcap_{\alpha\in\Lambda} B_\alpha] \subseteq \bigcap_{\alpha\in\Lambda} (B_\alpha] = \bigcap_{\alpha\in\Lambda} B_\alpha = B.$$

Therefore, $\bigcap_{\alpha\in\Lambda} B_\alpha$ is an ordered $\max\{m_\alpha \mid \alpha \in \Lambda\}$-bi-ideal of S.

Remark 2.15. Let $m \in \mathbb{N}$. Then every ordered m-left ideal of an ordered semiring S is an ordered m-bi-ideal of S. It is also true for the case of ordered m-right ideals.

Proof. Let L be an ordered m-left ideal of S. Then $LS^mL = L(S^mL) \subseteq LL \subseteq L$.

Remark 2.16. Let $m, n \in \mathbb{N}$. Then every ordered (m,n)-quasi-ideal of an ordered semiring S is an ordered $\max\{m,n\}$-bi-ideal of S.

Proof. Let A be an ordered (m,n)-quasi-ideal of S. It is not difficult to show that A is closed under multiplication and so it is sufficient to show that $AS^{\max\{m,n\}}A \subseteq A$. We assume that $m \geq n$. Then $AS^{\max\{m,n\}}A \subseteq AS^mS \subseteq AS^m \subseteq AS^n$ and $AS^{\max\{m,n\}}A \subseteq SS^mA \subseteq S^mA$. This implies that $AS^{\max(m,n)}A \subseteq S^mA \cap AS^n \subseteq (\Sigma S^mA] \cap (\Sigma AS^n] \subseteq A$. In case $m < n$, we can prove similarly.

The converse of the above remark is not true as show by the following example given by Munir and Shafiq (2018).

Example 2.17. Let $S = \{a, b, c, d\}$. Define binary operations $+$ and $\cdot$ on S by the following tables;

$+$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	b	b	b
c	c	b	b	b	b
d	d	b	b	b	b
e	e	b	b	b	c

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	b	b	b
d	a	b	b	b	c
e	a	b	b	c	c

Define a binary relation $\leq$ on S by $\leq := \{(a,a),(b,b),(c,c),(d,d),(e.e),(b,c),(b,d)\}$. Then $(S,+,\cdot,\leq)$ is an additively commutative ordered semiring with an absorbing zero a. Let $B = \{a,b,d\}$. We have that B is an ordered 1-bi-ideal of S, but not an ordered $(1,1)$-quasi-ideal of S.

Now, we describe the forms of the ordered m-left ideal, ordered m-right ideal, ordered (m,n)-quasi-ideal and ordered m-bi-ideal generated by a nonempty subset of an ordered semiring. Let A be a nonempty subset of an ordered semiring S and $m,n \in \mathbb{N}$. We denote the notations $L_m(A)$, $R_m(A)$, $Q_{(m,n)}(A)$ and $B_m(A)$ to be the ordered m-left ideal, ordered m-right ideal, ordered (m,n)-quasi-ideal and ordered m-bi-ideal of Sgenerated by A, respectively.

Theorem 2.18. Let A be a nonempty subset of an ordered semiring S and $m,n \in \mathbb{N}$. Then

(i) $L_m(A) = (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^mA]$;

(ii) $R_m(A) = (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma AS^m]$;

(iii) $Q_{(m,n)}(A) = \left(\Sigma A + \Sigma A^2 + \cdots + \Sigma A^{\max(m,n)} + \left((\Sigma S^mA] \cap (\Sigma AS^n]\right)\right]$.

Proof. We now show only case of (i) because (ii) and (iii) can be proved similarly.

Let A be a nonempty subset of an ordered semiring S and $m \in \mathbb{N}$ and set $L = (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A]$. It is clear that L is closed under addition and $L = (L]$. So, we must show that $A \subseteq L$, $L^2 \subseteq L$, $S^m L \subseteq L$ and if there is an ordered m-left ideal K of S such that $A \subseteq K$, then $L \subseteq K$.

Firstly, we have that $A \subseteq \Sigma A \subseteq \Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A \subseteq L$.

Secondly, we obtain that

$$\begin{aligned} L^2 &= (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A]^2 \\ &\subseteq ((\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A)(\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A)] \\ &\subseteq (\Sigma A^2 + \Sigma A^3 + \cdots + \Sigma A^{2m} + \Sigma S^m A^2 + \cdots + \Sigma S^m A^{m+1} + \Sigma A S^m A + \cdots + \Sigma A^m S^m A + \Sigma S^m A S^m A] \\ &\subseteq (\Sigma A^2 + \Sigma A^3 + \cdots + \Sigma A^m + \Sigma S^m A] \subseteq L. \end{aligned}$$

Thirdly, we consider

$$\begin{aligned} S^m L &= S^m(\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A] \\ &\subseteq (\Sigma S^m A + \Sigma S^m A^2 + \cdots + \Sigma S^m A^m + \Sigma S^m S^m A] \\ &\subseteq (\Sigma S^m A] \subseteq L. \end{aligned}$$

Finally, let K be an ordered m-left ideal of S such that $A \subseteq K$. Then, we can obtain that

$$\begin{aligned} L &= (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A] \\ &\subseteq (\Sigma K + \Sigma K^2 + \cdots + \Sigma K^m + \Sigma S^m K] \\ &\subseteq (\Sigma K + \Sigma K + \cdots + \Sigma K + \Sigma K] \\ &= (\Sigma K] = (K] = K. \end{aligned}$$

Therefore, $L = L_m(A)$.

Theorem 2.19. Let A be a nonempty subset of an ordered semiring S and $m \in \mathbb{N}$. Then

$$B_m(A) = (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A].$$

Proof. Let A be a nonempty subset of an ordered semiring S and $m \in \mathbb{N}$ and set $B = (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A]$. It is clear that B is closed under addition and $B = (B]$. So, we must show that $A \subseteq B$, $B^2 \subseteq B$, $BS^m B \subseteq B$ and if there is an ordered m-bi-ideal C of S such that $A \subseteq C$, then $B \subseteq C$.

Firstly, we have that $A \subseteq \Sigma A \subseteq \Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A \subseteq B$.

Secondly, we obtain that

$$\begin{aligned} B^2 &= (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A]^2 \\ &\subseteq ((\Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A)(\Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A)] \\ &\subseteq (\Sigma A^2 + \Sigma A^3 + \cdots + \Sigma A^{2m+2} + \Sigma A S^m A^2 + \cdots + \Sigma A S^m A^{m+2} + \Sigma A^2 S^m A + \cdots + \Sigma A^{m+2} S^m A + \\ &\qquad \Sigma A S^m A^2 S^m A] \\ &\subseteq (\Sigma A^2 + \Sigma A^3 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A] \\ &\subseteq B. \end{aligned}$$

Thirdly, we consider

$$\begin{aligned} BS^m B &= (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A] S^m (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A] \\ &\subseteq ((\Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A) S^m (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A)] \\ &\subseteq ((\Sigma A S^m + \Sigma A^2 S^m + \cdots + \Sigma A^{m+1} S^m + \Sigma A S^m A S^m)(\Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A)] \\ &\subseteq (\Sigma A S^m A + \Sigma A S^m A^2 + \cdots + \Sigma A S^m A^{m+1} + \Sigma A S^m A S^m A + \cdots + \Sigma A S^m A S^m A S^m A] \\ &\subseteq (\Sigma A S^m A] \subseteq B. \end{aligned}$$

Finally, let C be an ordered m-bi-ideal of S such that $A \subseteq C$. Then, we can obtain that

$$\begin{aligned} B &= (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A] \\ &\subseteq (\Sigma C + \Sigma C^2 + \cdots + \Sigma C^{m+1} + \Sigma C S^m C] \\ &\subseteq (\Sigma C + \Sigma C + \cdots + \Sigma C + \Sigma C] \\ &= (\Sigma C] = (C] = C. \end{aligned}$$

Therefore, $B = B_m(A)$.

We recall the following two lemmas which are given by Palakawong na Ayutthaya and Pibaljommee (2016) to be used in next theorems.

Lemma 2.20. An ordered semiring S is regular if and only if $R \cap L = (\Sigma RL]$ for every right ordered ideal R and every left ordered ideal L of S.

Lemma 2.21. An ordered semiring S is regular if and only if $B = (\Sigma BSB]$ for every ordered bi-ideal B of S.

Now, we give two interesting characterizations of regular ordered semirings as follows.

Theorem 2.22. An ordered semiring S is regular if and only if there exist $n, m \in \mathbb{N}$ such that $R \cap L = (\Sigma RL]$ for every ordered n-right ideal R and every ordered m-left ideal L of S.

Proof. $(\Rightarrow)$ It follows from Lemma 2.20 where we consider a right ordered ideal and a left ordered ideal as an ordered 1-right ideal and an ordered 1-left ideal of S, respectively.

$(\Leftarrow)$ Assume that there exist $n, m \in \mathbb{N}$ such that $R \cap L = (\Sigma RL]$ for every ordered n-right ideal R and every ordered m-left ideal L of S. Let $A \subseteq S$. By assumption and Remark 1.4, we have that

$$\begin{aligned} A &\subseteq R_n(A) \cap L_m(A) = (\Sigma R_n(A) L_m(A)] \\ &= \left(\Sigma(\Sigma A + \Sigma A^2 + \cdots + \Sigma A^n + \Sigma A S^n](\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A]\right] \\ &\subseteq (\Sigma(\Sigma A + \Sigma A^2 + \cdots + \Sigma A^n + \Sigma A S^n)(\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A)] \\ &\subseteq (\Sigma(\Sigma A^2 + \Sigma A^3 + \cdots + \Sigma A^{n+m} + \Sigma A S^n A + \cdots + \Sigma A S^n S^m A + \Sigma A S^m A + \cdots + \Sigma A S^n S^m A)] \\ &\subseteq (\Sigma A^2 + \Sigma A^3 + \cdots + \Sigma A^{n+m} + \Sigma A S^n A + \cdots + \Sigma A S^n S^m A + \Sigma A S^m A + \cdots + \Sigma A S^n S^m A] \\ &\subseteq (\Sigma A^2 + \Sigma A S A]. \end{aligned}$$

Now, $A \subseteq (\Sigma A^2 + \Sigma A S A]$. It leads to

$$\Sigma A^2 = \Sigma A A \subseteq \Sigma(A(\Sigma A^2 + \Sigma A S A]) \subseteq \Sigma(\Sigma A^3 + \Sigma A^2 S A] \subseteq \Sigma(\Sigma A S A] \subseteq (\Sigma A S A].$$

This implies that

$$A \subseteq (\Sigma A^2 + \Sigma A S A] \subseteq \left((\Sigma A S A] + \Sigma A S A\right] \subseteq \left((\Sigma A S A + \Sigma A S A]\right] = (\Sigma A S A].$$

By Remark 1.5, we obtain that S is regular.

Using the proof of Theorem 2.22, we can directly obtain the following corollary.

Corollary 2.23. An ordered semiring S is regular if and only if there exist $n, m \in \mathbb{N}$ such that $A \subseteq (\Sigma R_n(A) L_m(A)]$ for every $A \subseteq S$.

Theorem 2.24. An ordered semiring S is regular if and only if there exist $m \in \mathbb{N}$ such that $B = (\Sigma B S^m B]$ for every ordered m-bi-ideal B of S.

Proof. $(\Rightarrow)$ It follows from Lemma 2.21 where we consider an ordered bi-ideal as an ordered 1-bi-ideal of S.

$(\Leftarrow)$ Assume that there exists $m \in \mathbb{N}$ such that $B = (\Sigma B S^m B]$ for every ordered m-bi-ideal B of S. Let $A \subseteq S$. By assumption, we have that $A \subseteq B_m(A) = (\Sigma B_m(A) S^m B_m(A)]$. By Remark 2.15, we have that

$B_m(A) \subseteq R_m(A)$ and $B_m(A) \subseteq L_m(A)$. It turns out
$A \subseteq B_m(A) = (\Sigma B_m(A)S^m B_m(A)] \subseteq (\Sigma R_m(A)S^m L_m(A)] \subseteq (\Sigma R_m(A)L_m(A)]$. By Corollary 2.23, we can obtain that S is regular.

Discussion and Conclusion

In this work, we now have the notions of ordered m-left ideals, ordered m-right ideals, ordered (m,n)-quasi-ideals and ordered m-bi-ideals of ordered semirings which are generalizations of ordered ideals and some of their properties. Moreover, we have two interesting characterizations of regular ordered semirings by ordered m-left ideals, ordered m-right ideals and ordered m-bi-ideals which are extensions of results of Palakawong na Ayutthaya and Pibaljommee (2016). However, ordered (m,n)-quasi-ideals have not been used to characterize regular ordered semirings in this work and it will be studied them in the future works.

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